Can an ideal give you a maximal space?[1]

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Section **Topology**

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How to add a set A as open in topological space $\langle X, \tau \rangle$?

Simple way

$$\tau_A = \{\emptyset, A, X\}$$

Resulting topology is $\tau^A = \tau \vee \tau_A$, that is, the topology generated with the subbase

$$\tau \cup \{A\}$$

Question: Is there an ideal $\mathcal{I}_A$ such that $\tau^*(\mathcal{I}_A) = \tau^A$?[2]
Answer: In general: NO
Question: Is there an ideal $\mathcal{I}_A$ such that $A \in \tau^*(\mathcal{I}_A)$?
Answer: YES, $\mathcal{I}_A$ is $P(X)$ or the principal ideal containing $X \setminus A$ and its subsets.
Question: Is there an minimal ideal $\mathcal{I}_A$ such that $A \in \tau^*(\mathcal{I}_A)$?
Answer: Maybe, but, at least there exists not so large ideal.

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Local function

$\langle X, \tau \rangle$ - topological space

$$x \in \mathrm{Cl}(A) \Leftrightarrow \text{ for each } U \in \tau(x)\ A \cap U \notin \{\emptyset\}$$

$\mathcal{I}$ - an ideal on X

$$x \in A^*_{(\tau,\mathcal{I})} \Leftrightarrow \text{ for each } U \in \tau(x)\ A \cap U \notin \mathcal{I}$$

$\langle X, \tau, \mathcal{I} \rangle$ - ideal topological space [Kuratowski 1933]

$A^*_{(\tau,\mathcal{I})}$ (briefly A^*) - **local function**

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$A^*_{(\tau,\mathcal{I})}$ (briefly A^*) - **local function**

Local function and topology τ^*

For $\mathcal{I} = \{\emptyset\}$ we have that $A^*(\mathcal{I},\tau) = \mathrm{Cl}(A)$.
For $\mathcal{I} = P(X)$ we have that $A^*(\mathcal{I},\tau) = \emptyset$.
For $\mathcal{I} = Fin$ we have that $A^*(\mathcal{I},\tau)$ is the set of ω-accumulation points of A.
For $\mathcal{I} = \mathcal{I}_{count}$ we have that $A^*(\mathcal{I},\tau)$ is the set of condensation points of A.

$$A \subseteq B \Rightarrow A^* \subseteq B^* \qquad A^* = \mathrm{Cl}(A^*) \subseteq \mathrm{Cl}(A)$$

$$(A^*)^* \subseteq A^* \qquad (A \cup B)^* = A^* \cup B^*$$

If $I \in \mathcal{I}$, then $(A \cup I)^* = A^* = (A \setminus I)^*$.

$$\mathrm{Cl}^*(A) = A \cup A^*$$

$$\tau^*(\mathcal{I}) = \{A : \mathrm{Cl}^*(X \setminus A) = X \setminus A\}.$$

Set A is closed in τ^* iff $A^* \subseteq A$.

$$\tau \subseteq \tau^* = \tau^{**}$$

$\beta(\mathcal{I},\tau) = \{V \setminus I : V \in \tau, I \in \mathcal{I}\}$ is a basis for τ^*

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Topology τ^*

For $\mathcal{I} = \{\emptyset\}$ we have that $\tau^*(\mathcal{I}) = \tau$.
For $\mathcal{I} = P(X)$ we have that $\tau^*(\mathcal{I}) = P(X)$.
If $\mathcal{I} \subseteq \mathcal{J}$ then $\tau^*(\mathcal{I}) \subseteq \tau^*(\mathcal{J})$.
If $Fin \subseteq \mathcal{I}$ then $\langle X, \tau^* \rangle$ is T_1 space.
If $\mathcal{I} = Fin$, then $\tau^*_{ad}(\mathcal{I})$ is the cofinite topology on X.
If $\mathcal{I} = \mathcal{I}_{m0}$ - ideal of the sets of measure zero, then τ^*-Borel sets are precisely the Lebesgue measurable sets. (Scheinberg 1971)
For $\mathcal{I} = \mathcal{I}_{nwd}$ then $A^* = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$ and $\tau^*(\mathcal{I}_{nwd}) = \tau^\alpha$. ($\alpha$-open sets, $A \subseteq \mathrm{Int}(Cl(Int(A)))$. (Njástad 1965)

Making an ideal

For each $x \in A$ we will take its arbitrary neighbourhood U_x (van Douwen called this "a neighbourhood assignment").

$$\{U_x \setminus A : x \in A\}$$

will generate the ideal $\mathcal{I}_A$ on X.
Note: if $x \in \text{Int}(A)$, then there exists $U \in \tau(x)$ such that $U \setminus A = \emptyset$.
Therefore we will use the family

$$\{U_x \setminus A : x \in A \setminus \text{Int}(A)\}.$$

Theorem
If $(X, \tau, \mathcal{I}_A)$ is an ideal topological space, then $A \in \tau^*$.

Theorem (Smaller ideal)
Let (X, τ) be a T_1-space. If $A \notin \tau$, then there is no minimal ideal which generates in the ideal topological space $(X, \tau, \mathcal{I}_A)$ topology τ^* which contains A.

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Preserving regular open sets

Set U is regular open iff $U = \mathrm{Int}(\mathrm{Cl}(U))$
Family of regular open sets in τ generates topology τ_s

Theorem
If $\tau \cap \mathcal{I} = \{\emptyset\}$ then $\tau_s = (\tau^*)_s$.

Lemma
$\tau \cap \mathcal{I}_A = \{\emptyset\}$ if and only if for every $x \in A \setminus \mathrm{Int}(A)$, we have $\mathrm{Int}(U_x \setminus A) = \emptyset$.

Proposition
If $A \subseteq X$ is a dense set in τ, then regular open sets in τ are also regular open in τ^*, i.e. $\tau_s = (\tau^*)_s$.

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Preserving connectivity

Set A is preopen in a topological space (X, τ) iff $A \subseteq \text{Int}(\text{Cl}(A))$.
Every dense set is preopen

Proposition

If $(X, \tau, \mathcal{I}_A)$ is an ideal topological space and $A \subseteq X$ is not preopen in τ, then τ^* is not connected.

Remark

If $\tau \cap \mathcal{I}_A \neq \{\emptyset\}$, then (X, τ^*) is also not connected, since there is a set B open in τ (and therefore in τ^*) which is at the same time τ^*-closed, because $B \in \mathcal{I}_A$.

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Let us now modify the ideal $\mathcal{I}_A$ as follows.

Definition

Define the ideal $\mathcal{I}_A'$ as the ideal generated with the family of sets $\{U_x \setminus A : x \in A\}$, by:

1. if exists, let U_x be a minimal neighbourhood of $x \in X$;
2. if not, then we take, if possible, $U_x \in \tau(x)$ such that $U_x \subseteq \mathrm{Int}(\mathrm{Cl}(A))$;
3. if not, then we choose U_x arbitrary.

Lemma

$\tau \cap \mathcal{I}_A' = \{\emptyset\}$ if and only if A is a preopen set.

Theorem

If $A \subseteq X$ is a preopen set in τ, then $\tau_s = (\tau^*)_s$.

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In the topological space (X, τ) let $\mathcal{D}$ be the family of dense sets. We define a subfamily $\mathcal{D}' \subseteq \mathcal{D}$ with the following property: for every finite subfamily $\overline{\mathcal{D}}$ in $\mathcal{D}'$ the intersection $\bigcap \overline{\mathcal{D}}$ is dense in X. We will say that $\mathcal{D}'$ has **the dense finite intersection property**.

Precisely, we will consider the ideal $\mathcal{I}_D$ generated by the family of sets $\{C : C \subseteq (X \setminus D)\}$ for all $D \in \mathcal{D}'$, their subsets and their finite unions, where $\mathcal{D}'$ is a maximal family with the dense finite intersection property.

Lemma

If (X, τ) is an arbitrary topological space, then $\tau \cap \mathcal{I}_D = \{\emptyset\}$.

Proposition

For $(X, \tau, \mathcal{I}_D)$ holds $\tau_s = (\tau^*)_s$.

Thus, semiregularizations of τ and τ^* are the same.

Theorem

For the ideal topological space $(X, \tau, \mathcal{I}_D)$ we have that τ^* is the r.o. maximal topology on X, i.e. (X, τ^*) is a submaximal space.

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What about Maximal connected spaces?[3]

Unfortunately, using this method of ideals, in most of the cases we will not obtain maximal connected topology.

Space is maximal connected iff it is nearly maximal connected and every dense set is open.

The problem is that the property of nearly maximal connectedness only depends on regular-open sets, which implies that if starting topology τ is not nearly maximal connected, then also obtained τ^* will not have this property.

If we want to add new regular open sets, we will have to add some sets which are not preopen, but, in that case, we will lose connectivity.

So, to obtain maximal connected space with this method, we have to start with nearly maximal connected space. Example of such space are maximal singular expansions of real line.

On the other hand, natural topology on the real line is not nearly maximal connected, and therefore, by this method it can not be extended up to the maximal connected topology.

Njamcul, A., Pavlović, A., On topology expansion using ideals. arXiv:2312.03197 (2023). https://doi.org/10.48550/arXiv.2312.03197

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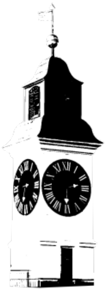

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